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ON THE NATURE OF ATTENUATION OF A FREE SURFACE ELEVATION CAUSED BY AN INITIAL PERTURBATION

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Asymptotics of solution for considerable times of the Cauchy-Poisson problem of waves induced by an initial perturbation at a point of the free surface of a viscous incompressible fluid of infinite depth is derived with the use of linearized Navier-Stokes equations. The solution is obtained in integral form by the method of successive integral transformations. Singular points of integrands are then determined, and their effect on the value of integrals is asymptotically calculated for considerable times. Limits of applicability of Sretenskii's integral as a solution of the considered problem in the case of simplified definition of waves [1] are determined.

1. The problem of waves induced at the free surface of a viscous incompressible fluid of infinite depth by an initial elevation is considered in linear formulation

$$\partial \mathbf{v}/\partial t = -\rho^{-1}\nabla p + \nu\Delta \mathbf{v}, \quad \operatorname{div} \mathbf{v} = 0, \quad p = p_1 + \rho g z \tag{1.1}$$

$$\zeta = \zeta_*, \quad \mathbf{v} = 0, \quad t = 0$$

$$-p + \rho g\zeta + 2\rho\nu\partial v_z/\partial z = 0, \quad \partial\zeta/\partial t = v_z \quad z = 0$$

$$\partial v_x/\partial z + \partial v_z/\partial x = 0, \quad \partial v_y/\partial z + \partial v_z/\partial y = 0$$

$$\mathbf{v} \to 0, \quad p \to 0 \quad x^2 + y^2 + z^2 \to \infty; \quad \partial \mathbf{v}/\partial x \to 0$$

$$d\mathbf{v}/dy \to 0 \quad x^2 + y^2 \to \infty$$

The coordinate origin is located at the (fluid) unperturbed surface and the Oz-axis is directed vertically upward.

2. Let us consider the problem of motion induced by an initial elevation of the free surface at the coordinate origin. We assume

$$\zeta_* = \lim_{b\to 0}\frac{S}{\pi}\frac{b}{b^2 + x^2}, \ b > 0; \ v_y \equiv 0, \ \frac{\partial \mathbf{v}}{\partial y} \equiv 0, \ \frac{\partial p}{\partial y} \equiv 0$$

where S is the area of the elevated fluid. By applying to (1.1) the integral Fourier transformation with respect to x and that of Laplace with respect to t, we obtain

$$\zeta = \lim_{b\to 0}\frac{S}{\pi}\int_0^\infty e^{-\xi b}\, X(t, \xi)\cos\xi x\, d\xi \tag{2.1}$$

$$X(t, \xi) = \frac{1}{2\pi i}\int_{s_0 - i\infty}^{s_0 + i\infty}\left[1 - \frac{1}{\lambda\Delta(s)}\right] e^{\nu\xi^2 ts}\frac{ds}{s} \tag{2.2}$$

$$s_0 > -1, \quad s_0 > \operatorname{Re} s_j, \quad \Delta(s_j) = 0$$

$$\Delta(s) = (s + 2)^2 - 4(s + 1)^{1/2} + \lambda^{-1}, \ \operatorname{Re}(s + 1)^{1/2} > 0, \quad \lambda = \nu^2\xi^3 g^{-1}$$

3. According to [2] (Sect. 349) the equation $\Delta(s) = 0$ has two roots $s_{1,2}$, which satisfy the condition $\mathrm{Re}\,(s+1)^{1/2} > 0$. These roots are complex conjugate for $\lambda < \lambda_*$ and negative for $\lambda > \lambda_*$ (see [1]). The value of λ_* is determined by the condition of multiplicity of root s or, what is the same, by the condition that root b of equation

$$(b^2+1)^2 - 4b + \lambda^{-1} = 0, \quad b = (s+1)^{1/2} \tag{3.1}$$

must be reiterated. This condition yields for the determination of λ_* the system

$$b_*(b_*^2+1) - 1 = 0, \quad (b_*^2+1)^2 - 4b_* + \lambda_*^{-1} = 0 \tag{3.2}$$

By the theorem on the roots of an algebraic equation (see [3], 6.3, Sect. 6, Chapt. 8) the root of Eq. (3.1) can be represented in the form

$$\lambda^{-1/4} \sum_{k=0}^{\infty} A_k \lambda^{1/4 k}, \quad \text{if} \quad |\lambda| < \lambda_*$$

$$\sum_{k=0}^{\infty} B_k \lambda^{-k}, \quad \text{if} \quad |\lambda| > \lambda_*$$

Substituting the series into Eq. (3.1) and collecting coefficients at like powers of λ, we determine A_k and B_k. Taking into account the condition $\mathrm{Re}\,(s+1)^{1/2} > 0$, we obtain the related formulas for s_j ($j = 1,2$) in the form of convergent series

$$s_j = \lambda^{-1/2} \sum_{k=0}^{\infty} a_{kj} \lambda^{1/4 k}, \quad |\lambda| < \lambda_*; \qquad s_j = \sum_{k=0}^{\infty} c_{kj} \lambda^{-k}, \quad |\lambda| > \lambda_* \tag{3.3}$$

$$a_{0j} = i\delta_j, \; a_{1j} = 0, \; a_{2j} = -2, \; a_{3j} = 2\exp\left(\frac{-\pi i}{4}\delta_j\right), \ldots; \; \delta_{1,2} = \pm 1$$

$$c_{01} = 0, \quad c_{11} = -1/2, \quad c_{21} = -3/16, \ldots; \; c_{02} \approx -0.9126; \ldots$$

To determine the integral $X(t, \xi)$ we slit, as in [4], along the negative real axis from $-\infty$ to -1, setting at the upper and lower sides of the slit

$$s = 1 + u^2 \pm i0, \quad (s+1)^{1/2} = \pm iu$$

Using the Cauchy theorem on residues and the previous substitution, from (2.2) we obtain

$$X(t,\xi) = \Phi(t,\xi) + \Psi(t,\xi); \; \Phi(t,\xi) = -\frac{8e^{-\nu\xi^2 t}}{\lambda\pi} \int_0^{\infty} \frac{e^{-\nu\xi^2 t u^2}}{\Delta_0(u)} \frac{u^2}{1+u^2} du \tag{3.4}$$

$$\Delta_0(u) = [(1-u^2)^2 + \lambda^{-1}]^2 + 16u^2; \qquad \Psi(t,\xi) = f_0(t,\xi) = -\lambda^{-1}\mathrm{Re}\, f(s_1)$$

$$|\lambda| < \lambda_*$$

$$f(s) = s^{-1}(s+1)^{1/2}[(s+2)(s+1)^{1/2} - 1]^{-1} \exp(\nu\xi^2 t s)$$

$$\Psi(t,\xi) = \sum_{j=1}^{2} f_j(t,\xi), \; f_j(t,\xi) = -(2\lambda)^{-1} f(s_j), \; |\lambda| > \lambda_*$$

where the values of $s_{1,2}$ for $|\lambda| > \lambda_*$ and s_1 for $|\lambda| < \lambda_*$ are those defined in (3.3). To determine ζ we subdivide the interval of integration with respect to ξ in (2.1) into three parts as follows: $[0, \xi_1]$, $[\xi_1, \xi_2]$, $[\xi_2, \infty)$; $\xi_1 < \xi_*$, $\xi_2 > \xi_*$, $\xi_* = g^{1/4}\nu^{-1/2}\lambda_*^{1/4}$. Then

$$\zeta = J_1 + J_2 + J_3 + J_4 + J_5 \tag{3.5}$$

$$J_1 = \frac{S}{\pi}\int_0^{\xi_1} f_0(t,\xi)\cos\xi x\,d\xi, \quad J_2 = \frac{S}{\pi}\int_0^{\infty} \Phi(t,\xi)\cos\xi x\,d\xi$$

$$J_3 = \lim_{b\to 0}\frac{S}{\pi}\int_{\xi_2}^{\infty} e^{-\xi b} f_1(t,\xi)\cos\xi x\,d\xi, \quad J_4 = \frac{S}{\pi}\int_{\xi_2}^{\infty} f_2(t,\xi)\cos\xi x\,d\xi \tag{3.6}$$

$$J_5 = \frac{S}{\pi}\int_{\xi_1}^{\xi_2} X(t,\xi)\cos\xi x\,d\xi$$

Functions $f_0, f_{1,2}$ and Φ appearing in integrals (3.6) are defined by related formulas (3.4) and function $X(t,\xi)$ by formula (2.2).

4. Let us examine the behavior of ζ for $T = tg^{2/5}\nu^{-1/5} \to \infty$ in the case when

$$\gamma = \nu g^2 t^5 x^{-4} = T^5 X^{-4} \to \infty, \quad X \neq 0; \quad X = xg^{1/5}\nu^{-2/5}$$

Several asymptotics were derived for the considered problem in [5] for $\gamma \to 0$

The asymptotic behavior of integrals of the form J_1 was investigated in [6] for $T \to \infty$. On the assumption that $T \to \infty$, $\gamma \to \infty$ and $\omega = gt^2/|x| = T^2/|X| \to \infty$ by suitable integration by parts we obtain for J_1

$$J_1 = \frac{S}{\pi g t^2}\left\{-2 + \frac{120}{\omega^2} - \frac{945\cdot 2^5}{\omega^4} + O\left(\frac{1}{\omega^6}\right) + \frac{576}{T^3}\left[1 + O\left(\frac{1}{\omega}\right) + O\left(\frac{1}{T^{3/4}}\right)\right]\right\} \tag{4.1}$$

Here and subsequently $O(m) = cm$, where $c = \text{const}$.

Integral J_2 is determined with the use of the identity in (4.2). The first term of the integral J_2 is calculated by formulas 3.466_2, $3{,}952_4$ and 6.315_3 of tables in [7]. The absolute value of the second term is estimated. We have

$$J_2 = -\frac{S}{\pi g t^2}\left\{\left[1 + \frac{6}{\varkappa} + \frac{24}{\varkappa^2}\right]e^{-1/4\varkappa} - \frac{24}{\varkappa^2}\right\} + R$$

$$\varkappa = \frac{x^2}{\nu t}, \quad |R| \leqslant \frac{Sg^{1/3}}{\pi^{3/2}\nu^{2/3}}\left(\frac{15}{T^{7/2}} + \frac{63\sqrt{\pi}}{T^5}\right)$$

$$\Delta_0^{-1} \equiv \lambda^2 - F(u,\lambda), \quad F(u,\lambda) = [2\lambda(1-u^2)^2 + \lambda^2(1-u^2)^4 + 16\lambda^2 u^2]\,\Delta_0^{-1}$$

which for $T \to \infty$ yields

$$J_2 = \frac{S}{\pi g t^2}\left[\frac{24}{\varkappa^2}\delta + O\left(\frac{1}{T^{3/2}}\right)\right] \tag{4.2}$$

$$\varkappa \to \infty;\ \delta = 1,\ \varkappa^{-2}T^{3/2} \to \infty;\ \delta = 0,\ \varkappa^{-2}T^{3/2} \leqslant c_0 < \infty$$

$$J_2 = -\frac{S}{2\pi g t^2}\left[\sum_{n=0}^{\infty}\frac{2n+1}{(n+2)\,n!}\left(-\frac{\varkappa}{4}\right)^n + O\left(\frac{1}{T^{3/2}}\right)\right]$$

$$\varkappa \to 0,\ x \neq 0,\ \varkappa^N T^{3/2} \to \infty,\ \varkappa^{N+1}T^{3/2} \leqslant c_0 < \infty$$

In computing J_3 we restrict the expansion of f_1 to the first term of the series

$$J_3 \sim \lim_{b\to 0}\frac{S}{\pi}\int_{\xi_2}^{\infty}\exp(-\xi b)\cos\xi x\,\exp\left(-\frac{gt}{2\nu\xi}\right)d\xi$$

We extend the integration interval to zero and subtract the respective integral. The value of the first is obtained by using formula 3.324_1 of tables in [7], while the second is estimated. We have

$$J_3 \sim \frac{S}{\pi |X|} \operatorname{Re}\{i\alpha K_1(\alpha)\} + R, \quad \alpha = [2T|X|]^{1/2} \exp\left(-\frac{\pi i}{4}\right) \tag{4.3}$$

$$|R| \leqslant \frac{Sg^{1/3}}{\pi \nu^{2/3}} c \exp\left(-\frac{T}{2c}\right), \quad 1.19813 < c \leqslant c_0 < \infty$$

where K_1 is the Bessel function of the imaginary argument. Using its asymptotic expansion ([7], $8{,}451_6$) we find that J_3 decreases exponentially with respect to T for $x \neq 0$ and $T \to \infty$. Subsequent terms in the expansion of f_1 do not materially affect the behavior of the integral J_3 for $T \to \infty$.

The absolute value of the integral J_4 is estimated as

$$|J_4| \leqslant Sg^{1/3}\nu^{-2/3}T^{-1/2}e^{-c_1T}C, \; C = \text{const}, \; c_1 = (1 - b_*^2)/4 \tag{4.4}$$

where b_* is the root of the first equation of system (3.2).

To estimate the integral J_5 we first estimate function $X(t, \xi)$. We change the variable of integration in the integral (2.2) by setting $s = s_0 + iv$. Since $\xi_1 \leqslant \xi \leqslant \xi_2$, all singular points of the integrand in (2.2) lie to the left of the straight line

$$s = \min\{\operatorname{Re} s_j(\lambda_1), s_j(\lambda_2)\}, \quad \lambda_j = \nu^2\xi^3{}_j g^{-1}, \quad j = 1,2$$

Hence it is possible to set in (2.2) $s_0 = -(4\lambda)^{-1}$. The estimate for $X(t, \xi)$ is obtained by integrating in (2.2) once by parts and setting $\exp(i\xi^2Tv) = du$. Estimating now the absolute value of J_5 we obtain

$$|J_5| < Sg^{1/3}\nu^{-2/3}T^{-1}\exp\left(-\frac{T}{4c}\right)M, \; M = \text{const}, \; c = \text{const} \tag{4.5}$$

Substituting (4.1) — (4.5) into (3.5) we obtain various asymptotics for ζ for $T \to \infty$

$$\zeta = \frac{S}{\pi g t^2}\left[-2 + \frac{24}{\varkappa^2}\delta + O\left(\frac{1}{T^{3/2}}\right)\right] \varkappa \to \infty \tag{4.6}$$

$$\zeta = -\frac{S}{\pi g t^2}\left[\frac{9}{4} + \frac{1}{2}\sum_{n=1}^{N}\frac{2n+1}{(n+2)\,n!}\left(-\frac{\varkappa}{4}\right)^n + O\left(\frac{1}{T^{3/2}}\right)\right] \tag{4.7}$$

$$\varkappa \to 0, \; \gamma \to \infty, \; \varkappa^{2N}/T^{-3} \to \infty, \; \varkappa^{2(N+1)}/T^{-3} \leqslant c_0 < \infty$$

or in dimensionless variables

$$\eta = \zeta g^{1/3}\nu^{-2/3}, \; Q = Sg^{2/3}\nu^{-4/3}, \; X = xg^{1/3}\nu^{-2/3}, \; T = tg^{2/3}\nu^{-1/3} \tag{4.8}$$

$$\eta = \frac{Q}{\pi T^2}\left[-2 + \frac{24}{\varkappa^2}\delta + O\left(\frac{1}{T^{3/2}}\right)\right] \tag{4.9}$$

$$T \to \infty, \; \varkappa \to \infty, \; \gamma \to \infty; \; \varkappa = \frac{X^2}{T}, \; \gamma = \frac{T^5}{X^4}$$

$$\delta = 1, \quad \varkappa^{-4}T^3 \to \infty; \quad \delta = 0, \quad \varkappa^{-4}T^3 \leqslant c_0 < \infty$$

$$\eta = -\frac{Q}{\pi T^2}\left[\frac{9}{4} + \frac{1}{2}\sum_{n=1}^{N}\frac{2n+1}{(n+2)\,n!}\left(-\frac{\varkappa}{4}\right)^n + O\left(\frac{1}{T^{3/2}}\right)\right] \tag{4.10}$$

$$T \to \infty, \quad \varkappa \to 0, \quad \gamma \to \infty,$$

$$X \neq 0, \quad \varkappa^{2N} T^3 \to \infty, \quad \varkappa^{2(N+1)} T^3 \leqslant c_0 < \infty$$

The first term of expansion (4. 6) for ζ is the same as the first term of a similar expansion (3. 7) of Sretenskii's integral in [6], proposed by him for the simplified definition of waves in the considered problem ((46) in [1]). Thus it follows from (4. 6), as well as from (48) and (49) in [1], (3.7), (3.17), (3.20) and (3.26) in [6], and from (4.25) and (4.27) in [5] that for $\nu t x^{-2} \to 0$ Sretenskii's integral defines the behavior of free surface elevation in the considered problem to within infinitely small quantities.

Formulas (4. 7) and (4. 10) define the final stage of wave attenuation for which Sretenskii's integral is inapplicable.

5. Similar computations in the case of three-dimensional motion of fluid induced by an initial elevation at the coordinate origin yield (in dimensionless variables)

$$\eta = -\frac{Q}{\pi^{3/2} T^{5/2}} \left[\frac{1}{5} + \sum_{k=1}^{K_1} \frac{k+1}{(2k+5)k!} \left(-\frac{\varkappa}{4} \right)^k + O\left(\frac{1}{T^3} \right) \right] + \tag{5. 1}$$

$$\frac{Q}{\pi T^4} \left[\frac{51}{8} + \frac{1}{2} \sum_{k=1}^{K_2} \frac{(2k+1)!!}{(k!)^2} \frac{k^2+k+3}{k+4} \left(-\frac{\varkappa}{8} \right)^k + O\left(\frac{1}{T^{3/2}} \right) \right]$$

$$T \to \infty,\ \varkappa \to 0,\ \gamma \to \infty;\ \varkappa = \frac{R^2}{T},\ \gamma = \frac{T^5}{R^4};\ \ \varkappa^{K_1} T^3 \to \infty,\ \varkappa^{K_1+1} T^3 \leqslant c_0 < \infty$$

$$\varkappa^{K_2} T^{3/2} \to \infty,\ \varkappa^{K_2+1} T^{3/2} \leqslant c_0 < \infty$$

$$\eta = \frac{Q\delta_1}{\pi R^5} \left\{ 18 + \frac{\delta_2}{(\pi\gamma)^{1/2}} \sum_{n=0}^{N} \frac{4n^2+8n+15}{(2n-5)\, n!} [(2n+1)!!]^2 \frac{1}{\varkappa^n} \right\} + \tag{5. 2}$$

$$\frac{Q}{\pi T^4} \left[6 + O\left(\frac{1}{T^{3/2}} \right) \right]$$

$$T \to \infty, \quad \varkappa \to \infty, \quad \gamma \to \infty; \quad \delta_1 = 1, \quad T^6 \varkappa^{-5} \to \infty; \quad \delta_1 = 0$$

$$T^6 \varkappa^{-5} \leqslant c_0 < \infty$$

$$\delta_2 = 1, \quad T\varkappa^{-1} \to \infty; \quad \delta_2 = 0, \quad T\varkappa^{-1} \leqslant c_0 < \infty, \quad T^{3/2} \varkappa^{-(N+3/2)} \to \infty$$

$$T^{3/2} \varkappa^{-(N+3/2)} \leqslant c_0 < \infty$$

where Q is the dimensionless volume of the elevated fluid, R is the dimensionless distance from the coordinate origin $R = r g^{1/3} \nu^{-2/3}$, and η and T are defined in (4. 8).

The comparison of formulas (4. 9) and (4. 10) with (5. 1) and (5. 2) shows that a three-dimensional elevation of the free surface is more rapidly attenuated with time than a plane one. It should be noted that in the three-dimensional case an integral similar to that of Sretenskii defines the behavior of the free surface elevation to within infinitely smalls, if in the considered problem

$$T^{11} R^{-10} = t^{11} \nu^3 g^4 r^{-10} = (\nu t / r^2)^3 (g t^2 / r)^4 \to 0$$

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APPLICATION OF THE REGULAR REPRESENTATION OF SINGULAR INTEGRALS TO THE SOLUTION OF THE SECOND FUNDAMENTAL PROBLEM OF THE THEORY OF ELASTICITY

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A regular representation is proposed for singular integrals present in integral equations of the second fundamental problem of elasticity theory. This representation is used to realize the successive approximations method in solving internal and external problems. Questions of constructing a computational scheme are discussed.

Use of potential theory apparatus permits reduction of the analysis of the fundamental boundary value problems of elasticity theory to integral equations [1]. To solve the second fundamental problem, Weil constructed regular integral equations of the second kind which generally possess eigenfunctions. Hence, their solution can be realized only after all the eigenfunctions of the adjoint equation have been determined, which is a complicated problem.

The application of a generalized elastic potential of a simple layer also reduces the mentioned boundary value problem to integral equations of the second kind. It is true these equations are not Fredholm equations in the classical form since their kernels have a second order polarity, and the corresponding integrals should be understood in the principal value sense. Consequently, the equations themselves are called singular. The equations mentioned possess quite favorable spectral properties. In the case of the external problem (we denote it by T_a) the equation is solvable for an arbitrary right-hand side. In the case of the internal problem (T_i), the equation is solvable when the right-hand side satisfies definite conditions but they agree with the conditions for existence of the solution of the initial problem of elasticity theory (the principal vector and the principal vector-moment of the external forces equal zero) and hence are assumed satisfied according to the formulation of the problem.

Each of the methods of solving the integral equations starts from the possibility of evaluating the integral terms for some representation of the required density. The associated difficulties are aggravated in solving singular, especially nonuniform, integral equations.

Questions of realizing the mechanical quadrature method in application to singular